

Enhancement of superconductivity by Anderson localization

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Influence of disorder on the temperature of superconducting transition (T_c) is studied within the σ -model renormalization group framework. Electron-electron interaction in particle-hole and Cooper channels is taken into account and assumed to be short-range. Two-dimensional systems in the weak localization and antilocalization regime, as well as systems near mobility edge are considered. It is shown that in all these regimes the Anderson localization leads to strong enhancement of T_c related to the multifractal character of wave functions.

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Soon after the development of the microscopic theory of superconductivity (SC) by Bardeen, Cooper, and Schrieffer (BCS) [1], the question of influence of disorder on SC attracted a great deal of attention. It was found [2, 3] that the diffusive motion of electrons does not affect essentially the temperature T_c of superconducting transition, i.e the mean free path does not enter the expression for T_c . This statement is conventionally called “Anderson theorem”.

Effects of disorder-induced Anderson localization [4] on SC were considered in Refs. [5, 6]. It was found that, within the BCS approach, the SC in a disordered system persists up to the localization threshold and even in the localized regime near the Anderson transition. Furthermore, Refs. [5, 6] came to the conclusion that the mean-field T_c in these regimes remains unaffected by disorder (i.e. the Anderson theorem holds). In a parallel line of research, it was discovered [7–9] that an interplay of long-range ($1/r$) Coulomb interaction and disorder leads to suppression of T_c . These ideas were put on the solid basis by Finkelstein [10, 11] who developed the σ -model renormalization-group (RG) formalism.

Recently, Feigelman *et al.* [12], motivated by a large body of experimental data [13], returned to the problem of interplay of disorder and SC. They found that the eigenfunction multifractality near the localization threshold strongly affects properties of a superconductor. Their remarkable finding is that T_c is dramatically enhanced: its dependence on the coupling constant is no more exponential (as in the conventional BCS solution) but rather of a power-law type. This result was obtained on the basis of the BCS-type self-consistency equation, with Cooper attraction being the only interaction included.

In this paper we reconsider the problem of interplay of SC and localization in the framework of the σ -model RG. We take into account the interaction in all channels (singlet and triplet particle-hole, and Cooper). The key assumption that distinguishes our work from Ref. [10]

is the short-range character of interaction that physically corresponds to a strong screening of the long-range Coulomb interaction. We consider first two-dimensional (2D) systems in the weak-localization (WL) and weak-antilocalization (WAL) regimes and find a strong enhancement of T_c by disorder. We show that this effect can be traced back to multifractality of wave functions. We then extend the analysis to the vicinity of Anderson transitions in 2D and 3D (three-dimensional) systems.

We begin by considering a 2D system of the orthogonal symmetry class (i.e. with preserved spin-rotation symmetry). In the σ -model formalism the problem is characterized by four running couplings: the dimensionless resistance $t = 2/\pi g$ (where g is the conductivity of the system in units of e^2/h) and three interaction constants Γ_i corresponding to singlet particle-hole (Γ_s), triplet particle-hole (Γ_t), and Cooper (Γ_c) channels [10]. The RG treatment yields a set of five coupled equations for these couplings and the field renormalization constant z . It is convenient to switch from Γ_i to normalized coupling constants $\gamma_i = \Gamma_i/z$ (typically, $\gamma_s, \gamma_c < 0$ and $\gamma_t > 0$); then the equation for z does not affect the remaining equations. The case of long-range Coulomb interaction corresponds to $\gamma_s = -1$.

Assuming weak short-range interaction, $|\gamma_i| \ll 1$, we have obtained the following RG system of equations (see Supporting Material [14]):

$$\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2\gamma_c^2 \end{pmatrix}, \quad (1)$$

$$dt/dy = t^2 \quad (2)$$

$$d \ln z / dy = (t/2)(\gamma_s + 3\gamma_t + 2\gamma_c). \quad (3)$$

Here $y = \ln L$, where L is the running RG length scale. We measure lengths in units of the microscopic scale where the RG (1)-(3) starts. All the equations are written to the leading order in $t, \gamma_i \ll 1$. The first term on the r.h.s. of Eq. (1) represents the effects of disorder on

interaction, while the second term is the Cooper renormalization of γ_c . The r.h.s. of Eq. (2) describes the WL effect. We discard the Altshuler-Aronov-type contribution in Eq. (2) (describing renormalization of disorder by interaction) since it is of higher order in γ_i .

Let us analyze the RG flow governed by Eqs. (1) and (2). Equation (2) decouples from the rest, yielding

$$t^{-1}(y) = t_0^{-1} - y, \quad (4)$$

where the subscript 0 refers to the bare value of the corresponding coupling. This is the usual WL behavior. In the absence of Eq. (1) it would imply that strong Anderson insulator emerges at the scale $y = t_0^{-1}$. We now turn to Eq. (1). Let us assume that the disorder is sufficiently strong compared to the interaction, so that $t_0 \gg |\gamma_{i,0}|$. Then at the initial portion of the RG flow we can neglect the Cooper renormalization term, which leaves us with a linear system of equations. The corresponding 3×3 matrix has two eigenvalues: a positive one, $\lambda = 2t$ and a doubly degenerate negative one, $\lambda' = -t$. Therefore, as RG starts to operate, the vector formed by three couplings quickly (at $y \sim 1$) approaches the eigenvector corresponding to λ , i.e. $-\gamma_s = \gamma_t = \gamma_c \equiv \gamma$. Projecting the system (1) onto this eigenvector, we get

$$d\gamma/dy = 2t\gamma - 2\gamma^2/3. \quad (5)$$

We have checked that the neglected contributions do not affect the results in any essential way [14].

We will assume the initial value $\gamma_0 = (-\gamma_{s,0} + 3\gamma_{t,0} + 2\gamma_{c,0})/6$ to be negative which will imply SC (or at least a tendency towards it) [15]. This is, in particular, the case when the dominant bare interaction is the Cooper attraction $\gamma_{c,0} < 0$. Solving then Eq. (5), we find [14] that there are two distinct situations. If $|\gamma_0| \ll t_0^2$, the resistance t reaches a value of order unity when the interaction is still weak. This means that when the scale is further increased (i.e. the temperature is lowered), the system becomes an insulator. On the other hand, if $|\gamma_0| \gg t_0^2$, the RG flow develops a superconducting instability (blow up of the interaction) at a scale where the resistance is still small, $t \ll 1$. This RG scale determines the temperature T_c of superconducting transition,

$$T_c \sim \exp\{-2/t_0\}, \quad (6)$$

which is much higher than the clean BCS value

$$T_c^{BCS} \sim \exp\{-1/|\gamma_{c,0}|\} \quad (7)$$

in the considered regime of sufficiently strong disorder, $t_0 \gg |\gamma_{i,0}|$. When the dimensionless resistance becomes smaller than the interaction, Eq. (6) crosses over into Eq. (7). We thus find that T_c shows a non-monotonous dependence on the disorder strength and gets strongly enhanced (by a parametrically large factor in the exponential) in the intermediate range of resistivities t_0 ,

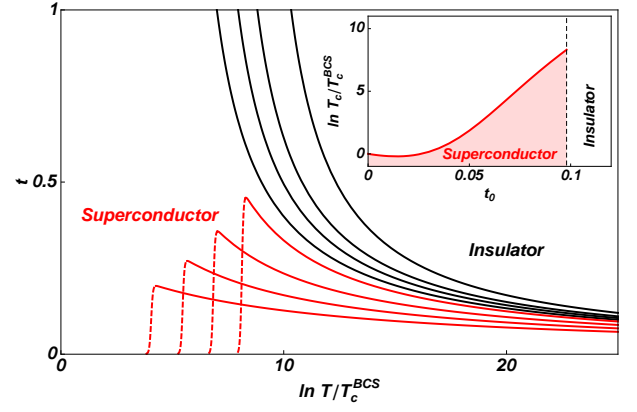


FIG. 1: (Color online) Temperature dependence of resistivity t near SIT in the 2D orthogonal symmetry class from numerical solution of Eqs. (1) and (2) for $\gamma_{s0} = -0.005$, $\gamma_{t0} = 0.005$, $\gamma_{c0} = -0.04$, and $t_0 = 0.065, 0.075, 0.085, 0.095, 0.10, 0.105, 0.11, 0.12$ (from bottom to top). The point where γ_c diverges determines T_c provided $t < 1$. Inset: dependence of T_c on the bare resistivity t_0 .

see Fig. 1. For given interaction strength, T_c is the largest when the system approaches the superconductor-insulator transition (SIT) that takes place at $t_0^2 \sim |\gamma_0|$.

It is important to emphasize a relation between this RG analysis of the interacting problem and the multifractality of wave functions in the non-interacting theory. The wave function (or, equivalently, local density of states) multifractality is a hallmark of criticality induced by Anderson localization [16]. It implies anomalous power-law scaling of moments (and, more generally, of correlation functions) of wave function amplitudes. In the field-theory language the corresponding exponents are scaling dimensions of composite operators. In the fourth order in wave function amplitudes ψ , which corresponds to the second order with respect to the σ -model field $Q \sim \psi\psi^\dagger$, there are two such operators. The dominant one has a negative scaling dimension (i.e. it is RG relevant), $\Delta_2 < 0$, which is the most famous representative of the family of anomalous dimensions Δ_q describing the wave function multifractality spectrum [16–18]. The second operator has a positive dimension $\mu_2 > 0$, which means that it is RG-irrelevant. It is worth mentioning that, despite the RG-irrelevant character, μ_2 controls the scaling behavior at Anderson transitions with short range interaction [19]. In terminology of Wegner who pioneered the Anderson-localization multifractal analysis [17], these exponents are denoted as $x_{2s} < 0$ and $x_{2a} > 0$, respectively. To the linear order in couplings γ_i , the RG equations of the interacting theory should be controlled by scaling dimensions at the non-interacting fixed point. We have verified that this is indeed the case. Specifically, the exponent $\lambda = 2t$, which is the positive eigenvalue of the matrix in (1) and which shows up in Eq. (5), is nothing but the anomalous fractal dimension (with an opposite

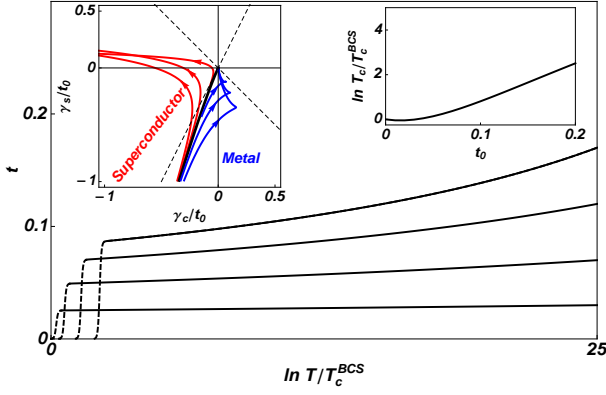


FIG. 2: (Color online) Temperature dependence of resistivity t in the 2D symplectic symmetry class from numerical solution of Eqs. (8)-(9) for $\gamma_{s0} = -0.005$, $\gamma_{c0} = -0.04$, and $t_0 = 0.02, 0.07, 0.12, 0.17$ (from bottom to top). Left inset: projection of RG flow (8)-(9) on a plane $t = \text{const}$. Separatrix (thick black curve) approaches the line $\gamma_s = 2\gamma_c$. Right inset: dependence of T_c on the bare resistivity t_0 .

sign), $\lambda = -\Delta_2$, while the second eigenvalue is the irrelevant exponent, $\lambda' = -t = -\mu_2$. Thus, the enhancement of T_c is intimately related to multifractality.

Let us consider now a 2D system with strong spin-orbit interaction. In this case the spin-rotation symmetry is broken and the system belongs to the symplectic symmetry class. The change of the symmetry leads to two important modifications of RG equations: (i) WAL replaces WL, and (ii) triplet interaction channel gets suppressed and can be discarded. The equations for the remaining interaction constants γ_s, γ_c and resistivity t read

$$\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 2\gamma_c^2 \end{pmatrix}, \quad (8)$$

$$dt/dy = -t^2/2. \quad (9)$$

The solution of Eq. (9) describes the WAL flow

$$t^{-1}(y) = t_0^{-1} + y/2. \quad (10)$$

The eigenvalues of the linear part of the system (8) are again related to the multifractality of wave functions (this time for the symplectic symmetry class): $\lambda = t/2 = -\Delta_2$; $\lambda' = -t = -\mu_2$. The dominant eigenvalue λ corresponds to the direction $-\gamma_s = \gamma_c \equiv \gamma$, see Fig. 2. Projecting the system onto this eigenvector, we get

$$d\gamma/dy = (t/2)\gamma - (4/3)\gamma^2. \quad (11)$$

Assuming $\gamma_0 = (-\gamma_{s,0} + 2\gamma_{c,0})/3 < 0$ and solving Eq. (11), we find the scale at which the coupling $|\gamma|$ becomes unity. This gives the transition temperature

$$T_c \sim \exp\{-\mathcal{C}/(t_0|\gamma_0|)^{1/2}\}, \quad (12)$$

where a numerical prefactor $\mathcal{C} \sim 1$ depends on the ratio $\gamma_{c,0}/\gamma_{s,0}$. Equation (12) is valid for $|\gamma_0| \ll t_0$; in the

opposite case the clean BCS result (7) is restored. The enhancement of T_c in Eq. (12) is again due to multifractality represented by the eigenvalue $\lambda = t/2 = -\Delta_2$ in Eq. (11). The enhancement is less efficient as compared to the orthogonal symmetry class, Eq. (6), because of antilocalizing behavior that leads to decrease of t and thus to weakening of multifractality (see Fig. 2).

We consider now a system at the Anderson transition point. This may be either a 2D or 3D symplectic class-system, or a 3D system of orthogonal symmetry. In all the cases, after the initial (fast) part of the RG evolution, where the Cooper term γ_c^2 is unimportant, γ_s and (in the orthogonal case) γ_t “adjust” to γ_c according to $\gamma_s = -\gamma_t = -\gamma_c$ (orthogonal) or $\gamma_s = -\gamma_c$ (symplectic). So, the main part of the RG evolution can be described by a single equation, in analogy with Eqs. (5) and (11):

$$d\gamma/dy = -\Delta_2\gamma - a\gamma^2, \quad a \sim 1, \quad (13)$$

where $\Delta_2 < 0$ is the fractal exponent for the given transition point. In particular, $\Delta_2 = -1.7 \pm 0.05$ for the 3D orthogonal-class and $\Delta_2 = -0.344 \pm 0.004$ for the 2D symplectic-class Anderson transitions [16]. The SC will take place if $\gamma_0 < 0$. Analyzing Eq. (13), we find

$$T_c^* \sim |\gamma_0|^{d/|\Delta_2|}. \quad (14)$$

for the transition temperature in d spatial dimensions. We see that at the Anderson transition point the enhancement of the superconducting T_c becomes even stronger than in 2D: T_c is now a power-law (rather than exponential) function of the interaction constant. Equation (14) agrees with the result of Ref. [12].

The RG analysis allows us also to analyze the situation when the system is slightly off the Anderson transition. It is convenient to characterize the distance to the critical point t_* by the correlation (localization) length $\xi \sim |t_0 - t_*|^{-\nu}$. The corresponding energy scale is the one-particle level spacing in the correlation volume $\delta_\xi \propto 1/\xi^d$. The result (14) retains its validity in a vicinity of t_* as long as $\delta_\xi \lesssim T_c$. On the insulating side ($t_0 > t_*$) the condition $\delta_\xi \sim T_c$ determines the point of the superconductor-insulator quantum phase transition (see Fig. 3) [20]. On the metallic side ($t_0 < t_*$), there is a crossover regime extending from $\delta_\xi \sim T_c^*$ to $\delta_\xi \sim 1$ that provides a matching between the result (14) at the Anderson-transition critical point and the clean BCS result [14].

We close the paper with several comments:

1) The true superconducting transition in 2D is of Berezinskii-Kosterlitz-Thouless (BKT) character, whereas we have calculated the mean-field transition temperature T_c . It was found however [21] that the corresponding temperatures do not differ much, $T_{\text{BKT}} \simeq T_c$. Therefore, our result that shows an exponential enhancement of T_c is expected to hold for T_{BKT} as well.

2) The key assumption of the above theory was the neglect of long-range ($1/r$) Coulomb interaction. We

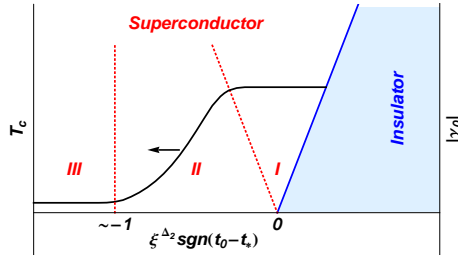


FIG. 3: (Color online) Schematic phase diagram in the interaction–disorder plane near the critical point. Solid (blue) line denotes SIT. Dependence of T_c on the distance from t_* at fixed value of γ_0 is shown by solid (black) curve. In the quantum critical regime (region I) T_c is given by Eq. (14). Away from criticality at $t < t_*$ (region III) the BCS expression (7) holds. In the crossover regime (region II) $T_c \sim \xi^{-3} \exp(-c_3 \xi^{\Delta_2} / |\gamma_{c,0}|)$ in 3D and $T_c \sim \xi^{-2} \exp(-c_2 \sqrt{\xi^{\Delta_2} / |\gamma_{c,0}|})$ for the symplectic symmetry class in 2D [14].

can think of the following situations when this should be justified: (i) a 3D material with a large background dielectric constant ϵ ; (ii) a 2D system on a substrate (or between two dielectrics) with large ϵ ; (iii) a 2D system with a nearby screening metallic layer; (iv) a system of interacting neutral fermions (i.e. cold atoms).

3) A natural question is whether this effect has already been observed in experiment. While one does see some enhancement of SC by disorder in several materials, one needs to argue that the Coulomb interaction is suppressed in order to attribute the increase of T_c to the effect of localization. In particular, a non-monotonous dependence of T_c on the normal-state resistivity was found in Refs. [22, 23] in structures for which large ϵ is expected.

To summarize, we have developed a σ -model RG theory describing interplay of SC and Anderson localization in a disordered system with short-range interactions. This theory predicts a strong enhancement of SC by Anderson localization in 2D systems (at intermediate disorder) and near localization transitions, implying a strongly non-monotonous dependence of T_c on normal-state resistivity. Remarkably, the localization physics responsible for increase of resistivity and thus driving the system towards an insulating state favors at the same time the SC. It remains to be seen whether this mechanism may be employed in practice to obtain structures with strongly enhanced T_c . The key condition is a suppression of the long-range component of the Coulomb interaction.

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SUPPLEMENTARY INFORMATION

I. ONE-LOOP RENORMALIZATION IN THE FINKEL'STEIN NLSM

A. Nonlinear σ -model: Definitions

The action of the Non-Linear Sigma Model (NLSM) is given as a sum of the non-interacting part, S_σ , and contributions arising from the interaction in the particle-hole singlet, $S_{\text{int}}^{(\rho)}$, particle-hole triplet, $S_{\text{int}}^{(\sigma)}$, and particle-particle (Cooper), $S_{\text{int}}^{(c)}$, channels [S1,S2]:

$$S = S_\sigma + S_{\text{int}}^{(\rho)} + S_{\text{int}}^{(\sigma)} + S_{\text{int}}^{(c)}, \quad (15)$$

where

$$S_\sigma = -\frac{g}{32} \int d\mathbf{r} \text{Tr}(\nabla Q)^2 + 4\pi T z \int d\mathbf{r} \text{Tr} \eta(Q - \Lambda) \quad (16)$$

$$S_{\text{int}}^{(\rho)} = -\frac{\pi T}{4} \Gamma_s \sum_{\alpha,n} \sum_{r=0,3} \int d\mathbf{r} \text{Tr} [I_n^\alpha t_{r0} Q] \text{Tr} [I_{-n}^\alpha t_{r0} Q] \quad (17)$$

$$S_{\text{int}}^{(\sigma)} = -\frac{\pi T}{4} \Gamma_t \sum_{\alpha,n} \sum_{r=0,3} \sum_{j=1}^3 \int d\mathbf{r} \text{Tr} [I_n^\alpha t_{rj} Q] \text{Tr} [I_{-n}^\alpha t_{rj} Q] \quad (18)$$

$$S_{\text{int}}^{(c)} = -\frac{\pi T}{2} \Gamma_c \sum_{\alpha,n} \sum_{r=0,3} (-1)^r \int d\mathbf{r} \text{Tr} [I_n^\alpha t_{r0} Q I_n^\alpha t_{r0} Q] \quad (19)$$

Here g is the total Drude conductivity (in units e^2/h and including spin) and we use the following matrices

$$\Lambda_{nm}^{\alpha\beta} = \text{sgn } n \delta_{nm} \delta^{\alpha\beta} t_{00}, \quad \eta_{nm}^{\alpha\beta} = n \delta_{nm} \delta^{\alpha\beta} t_{00}, \quad (I_k^\gamma)_{nm}^{\alpha\beta} = \delta_{n-m,k} \delta^{\alpha\beta} \delta^{\alpha\gamma} t_{00} \quad (20)$$

with α, β standing for replica indices and n, m corresponding to Matsubara fermionic energies $\epsilon_n = \pi T(2n + 1)$.

The matrices

$$t_{rj} = \tau_r \otimes s_j \quad (21)$$

operate in the particle-hole (index r) and spin (index j) spaces, with the corresponding Pauli matrices denoted by

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (22)$$

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (23)$$

The matrix field $Q(\mathbf{r})$ (as well as the trace Tr) acts in the replica, Matsubara, spin, and particle-hole spaces. It obeys the following constraints:

$$Q^2 = 1, \quad \text{Tr } Q = 0, \quad Q^\dagger = C^T Q^T C, \quad (24)$$

where

$$C = it_{12}, \quad C^T = -C. \quad (25)$$

It can be useful to represent Q as $Q = T^{-1} \Lambda T$, with matrices T obeying

$$CT^* = -TC, \quad (T^{-1})^* C = -CT^{-1}. \quad (26)$$

In order to avoid notational confusion, it is instructive to compare our notation with that of the reviews by Finkel'stein [S1] and by Belitz and Kirkpatrick [S2]. First of all, these authors use different definitions of Pauli matrices:

$$\text{Ref. [S1]} : \quad \tau_0^F = \tau_0, \quad \tau_j^F = i\tau_j, \quad \sigma_0^F = s_0, \quad \sigma_j^F = s_j, \quad j = 1, 2, 3 \quad (27)$$

$$\text{Ref. [S2]} : \quad \tau_0^{BK} = \tau_0, \quad \tau_j^{BK} = i\tau_j, \quad s_0^{BK} = s_0, \quad s_j^{BK} = -is_j, \quad j = 1, 2, 3 \quad (28)$$

The interaction terms (17), (18) and (19) coincide with terms in Eqs. (3.9a), (3.9b), and (3.9b) in Ref. [S1] with the coupling constants

$$\Gamma_s = -\frac{\pi\nu}{4}Z, \quad \Gamma_t = \frac{\pi\nu}{4}\Gamma_2, \quad \Gamma_c = \frac{\pi\nu}{4}\Gamma_c, \quad (29)$$

and with terms in Eqs. (3.92d), (3.92e), and (3.92f) in Ref. [S2] with

$$\Gamma_s = K^{(1)}, \quad \Gamma_t = K^{(2)}, \quad \Gamma_c = K^{(3)}/2, \quad (30)$$

where ν is the thermodynamic density of states including spin. Finally, the parameters g and z in Eq. (16) are related by $g = 4\pi\nu D$ and $z = (\pi\nu/4)Z$ to the corresponding parameters introduced in Ref. [S1] and by $g = 16/G$ and $z = H/2$ to those in Ref. [S2]. Note that Ref. [S1] focuses on the case of unscreened (long-ranged) Coulomb interaction: hence the interaction amplitude Γ_s in the singlet particle-hole channel is expressed through the frequency renormalization factor Z there. In our case of a short-range interaction, these quantities are independent variables.

B. Perturbation expansion

We shall use the square-root parametrization

$$Q = W + \Lambda\sqrt{1 - W^2}, \quad W = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}. \quad (31)$$

We adopt the following notations: $W_{n_1 n_2} = w_{n_1 n_2}$ and $W_{n_2 n_1} = \bar{w}_{n_2 n_1}$ with $n_1 > 0$ and $n_2 \leq 0$. The blocks (in Matsubara space) obey

$$\bar{w} = -Cw^T C, \quad w = -Cw^* C. \quad (32)$$

The second equality implies that in the expansion $w_{n_1 n_2}^{\alpha\beta} = \sum_{rj} (w_{n_1 n_2}^{\alpha\beta})_{rj} t_{rj}$ some of the elements $(w_{n_1 n_2}^{\alpha\beta})$ are real and some are purely imaginary.

Expanding the action S_σ to the second order in W , we find

$$S_\sigma^{(2)} = -\frac{g}{4} \sum_{rj} \sum_{n_1 n_2} \sum_{\alpha\beta} \int \frac{d\mathbf{p}}{(2\pi)^d} \left[p^2 + \frac{32\pi T z}{g} n_{12} \right] [w_{n_1 n_2}^{\alpha\beta}]_{rj} [\bar{w}_{n_2 n_1}^{\beta\alpha}]_{rj} \quad (33)$$

Hence, the propagator becomes

$$\langle [w_{n_1 n_2}^{\alpha_1 \beta_1}]_{r_1 j_1} [\bar{w}_{n_4 n_3}^{\beta_2 \alpha_2}]_{r_2 j_2} \rangle = \frac{2}{g} D_p(n_{12}) \delta^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} \delta_{n_1 n_3} \delta_{n_2 n_4} \delta_{r_1 r_2} \delta_{j_1 j_2}, \quad (34)$$

$$D_p^{-1}(n_{12}) = p^2 + \frac{32\pi T z}{g} n_{12}. \quad (35)$$

Note that $\langle ww \rangle$ and $\langle \bar{w}\bar{w} \rangle$ are also non-zero. From Eq. (35) we find

$$\langle \text{Tr } a w \text{ Tr } b \bar{w} \rangle = \frac{32}{g} \sum_{n_1 n_2} \sum_{\alpha\beta} \int \frac{d\mathbf{p}}{(2\pi)^d} D_p(n_{12}) \sum_{rj} [a_{n_2 n_1}^{\alpha\beta}]_{rj} [b_{n_1 n_2}^{\beta\alpha}]_{rj} \approx 4Y \text{Tr} \frac{1-\Lambda}{2} a \frac{1+\Lambda}{2} b, \quad (36)$$

$$Y = \frac{2}{g} \int \frac{d\mathbf{p}}{(2\pi)^d} D_p(0). \quad (37)$$

In what follows we will use the following identity:

$$\langle \text{Tr } A W \text{ Tr } B W \rangle = 2Y \text{Tr} \left[AB - \Lambda A \Lambda B - A C B^T C + \Lambda A \Lambda C B^T C \right]. \quad (38)$$

C. Comments on interaction in the Cooper channel

The Cooper channel interaction term $S_{\text{int}}^{(c)} = (-\pi T \Gamma_c / 2) O^{(c)}$ can be rewritten as

$$O^{(c)} = \frac{1}{2} \sum_{\alpha, n} \sum_{r=1,2} \sum_{j=0}^3 \text{Tr} [t_{rj} L_n^\alpha Q] \text{Tr} [t_{rj} L_n^\alpha Q], \quad (L_n^\alpha)^{\beta\gamma}_{km} = \delta_{k+m, n} \delta^{\alpha\beta} \delta^{\alpha\gamma} t_{00} \quad (39)$$

However, $\text{Tr}[t_{rj}L_n^\alpha Q] = 0$ for $j = 1, 2, 3$ since

$$\text{Tr}[t_{rj}L_n^\alpha Q] = \text{Tr}[t_{rj}L_n^\alpha Q]^T = -\text{Tr}[Ct_{rj}^T CL_n^\alpha Q] = -\text{Tr}[Ct_{rj}^* CL_n^\alpha Q] = \begin{cases} j = 0, & \text{Tr}[t_{r0}L_n^\alpha Q] \\ j = 1, 2, 3 & -\text{Tr}[t_{rj}L_n^\alpha Q] \end{cases}. \quad (40)$$

Therefore, the operator $O^{(c)}$ describing the interaction in the Cooper channel is fully determined by the Cooper-singlet channel:

$$O^{(c)} = \frac{1}{2} \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[t_{r0}L_n^\alpha Q] \text{Tr}[t_{r0}L_n^\alpha Q]. \quad (41)$$

D. Background field renormalization

1. Singlet interaction in particle-hole channel: $S_{\text{int}}^{(\rho)} = (-\pi T \Gamma_s / 4) O^{(\rho)}$

Performing transformation $Q \rightarrow T_0^{-1} Q T_0$, we find with the help of Eq. (38) and (26):

$$\begin{aligned} O^{(\rho)} &= \sum_{\alpha, n} \sum_{r=0,3} \text{Tr}[I_n^\alpha t_{r0} Q] \text{Tr}[I_{-n}^\alpha t_{r0} Q] \rightarrow \left\langle \sum_{\alpha, n} \sum_{r=0,3} \text{Tr}[I_n^\alpha t_{r0} T_0^{-1} Q T_0] \text{Tr}[I_{-n}^\alpha t_{r0} T_0^{-1} Q T_0] \right\rangle \\ &= \left\langle \sum_{\alpha, n} \sum_{r=0,3} \text{Tr}[T_0 I_n^\alpha t_{r0} T_0^{-1} W] \text{Tr}[T_0 I_{-n}^\alpha t_{r0} T_0^{-1} W] \right\rangle = 2Y \sum_{\alpha, n} \sum_{r=0,3} \text{Tr}[-I_n^\alpha t_{r0} Q_0 I_{-n}^\alpha t_{r0} Q_0 + I_n^\alpha t_{r0} Q_0 I_n^\alpha C t_{r0}^* C Q_0] \\ &= -2Y \sum_{\alpha, n} \sum_{r=0,3} \left\{ \text{Tr}[I_n^\alpha t_{r0} Q_0 I_{-n}^\alpha t_{r0} Q_0] + (-1)^r \text{Tr}[I_n^\alpha t_{r0} Q_0 I_n^\alpha t_{r0} Q_0] \right\}. \end{aligned} \quad (42)$$

Next, we represent the resulting expressions in terms of the operators entering in the interaction-induced part of the action, (17), (18), and (19):

$$\sum_{\alpha, n'} \sum_{r=0,3} \text{Tr}[I_{n'}^\alpha t_{r0} Q_0 I_{-n'}^\alpha t_{r0} Q_0] = \frac{1}{2} \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=0}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0] \text{Tr}[I_{-n}^\alpha t_{rj} Q_0] = \frac{1}{2} O^{(\rho)} + \frac{1}{2} O^{(\sigma)} \quad (43)$$

Finally, we see that the renormalized particle-hole singlet term is expressed through all three interaction operators :

$$O^{(\rho)} \rightarrow O^{(\rho)} - Y O^{(\rho)} - Y O^{(\sigma)} - 2Y O^{(c)}. \quad (44)$$

2. Triplet interaction in particle-hole channel: $S_{\text{int}}^{(\sigma)} = (-\pi T \Gamma_t / 4) O^{(\sigma)}$

Performing transformation $Q \rightarrow T_0^{-1} Q T_0$, we find with the help of Eq. (38) and (26):

$$\begin{aligned} O^{(\sigma)} &= \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} Q] \text{Tr}[I_{-n}^\alpha t_{rj} Q] \rightarrow \left\langle \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} T_0^{-1} Q T_0] \text{Tr}[I_{-n}^\alpha t_{rj} T_0^{-1} Q T_0] \right\rangle \\ &= \left\langle \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=1}^3 \text{Tr}[T_0 I_n^\alpha t_{rj} T_0^{-1} W] \text{Tr}[T_0 I_{-n}^\alpha t_{rj} T_0^{-1} W] \right\rangle \\ &= 2Y \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=1}^3 \text{Tr}[-I_n^\alpha t_{rj} Q_0 I_{-n}^\alpha t_{rj} Q_0 + I_n^\alpha t_{rj} Q_0 I_n^\alpha C t_{rj}^* C Q_0] \\ &= -2Y \sum_{\alpha, n} \sum_{r=0,3} \sum_{j=1}^3 \left\{ \text{Tr}[I_n^\alpha t_{rj} Q_0 I_{-n}^\alpha t_{rj} Q_0] - (-1)^r \text{Tr}[I_n^\alpha t_{rj} Q_0 I_n^\alpha t_{rj} Q_0] \right\}. \end{aligned} \quad (45)$$

Next,

$$\begin{aligned} \sum_{\alpha, n'} \sum_{r=0,3} \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0 I_{-n}^\alpha t_{rj} Q_0] &= \frac{1}{2} \sum_{\alpha, n} \sum_{r=0,3} \left\{ 3 \text{Tr}[I_n^\alpha t_{r0} Q_0] \text{Tr}[I_{-n}^\alpha t_{r0} Q_0] - \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0] \text{Tr}[I_{-n}^\alpha t_{rj} Q_0] \right\} \\ &= \frac{3}{2} O^{(\rho)} - \frac{1}{2} O^{(\sigma)}, \end{aligned} \quad (46)$$

and

$$\begin{aligned} \sum_{\alpha, n'} \sum_{r=0,3} (-1)^r \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0 I_n^\alpha t_{rj} Q_0] &= \frac{1}{2} \sum_{\alpha, n} \sum_{r=1,2} \left\{ 3 \text{Tr}[L_n^\alpha t_{r0} Q_0] \text{Tr}[L_n^\alpha t_{r0} Q_0] - \sum_{j=1}^3 \text{Tr}[L_n^\alpha t_{rj} Q_0] \text{Tr}[L_n^\alpha t_{rj} Q_0] \right\} \\ &= \frac{3}{2} \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[L_n^\alpha t_{r0} Q_0] \text{Tr}[L_n^\alpha t_{r0} Q_0] = 3O^{(c)}. \end{aligned} \quad (47)$$

Finally, we find

$$O^{(\sigma)} \rightarrow O^{(\sigma)} - 3YO^{(\rho)} + YO^{(\sigma)} + 6YO^{(c)}. \quad (48)$$

3. *Interaction in Cooper channel:* $S_{\text{int}}^{(c)} = (-\pi T \Gamma_c / 2) O^{(c)}$

Performing transformation $Q \rightarrow T_0^{-1} Q T_0$, we find with the help of Eq. (38) and (26):

$$\begin{aligned} O^{(c)} &= \sum_{\alpha, n} \sum_{r=0,3} (-1)^r \text{Tr}[I_n^\alpha t_{r0} Q I_{-n}^\alpha t_{r0} Q] = \frac{1}{2} \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[L_n^\alpha t_{r0} Q] \text{Tr}[L_n^\alpha t_{r0} Q] \\ &\rightarrow \frac{1}{2} \left\langle \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[L_n^\alpha t_{r0} T_0^{-1} Q T_0] \text{Tr}[L_n^\alpha t_{r0} T_0^{-1} Q T_0] \right\rangle \\ &= \frac{1}{2} \left\langle \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[T_0 L_n^\alpha t_{r0} T_0^{-1} W] \text{Tr}[T_0 L_n^\alpha t_{r0} T_0^{-1} W] \right\rangle = Y \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[-L_n^\alpha t_{r0} Q_0 L_n^\alpha t_{r0} Q_0 + L_n^\alpha t_{r0} Q_0 L_n^\alpha C t_{r0}^* C Q_0] \\ &= -2Y \sum_{\alpha, n} \sum_{r=1,2} \text{Tr}[L_n^\alpha t_{r0} Q_0 L_n^\alpha t_{r0} Q_0]. \end{aligned} \quad (49)$$

Next,

$$\begin{aligned} \sum_{\alpha, n'} \sum_{r=1,2} \text{Tr}[L_n^\alpha t_{r0} Q_0 L_n^\alpha t_{r0} Q_0] &= \frac{1}{2} \sum_{\alpha, n} \sum_{r=0,3} (-1)^r \sum_{j=0}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0] \text{Tr}[I_n^\alpha t_{rj} Q_0] \\ &= -\frac{1}{2} \sum_{\alpha, n} \sum_{r=0,3} (-1)^r \sum_{j=0}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0] \text{Tr}[I_{-n}^\alpha C t_{rj}^* C Q_0] \\ &= -\frac{1}{2} \sum_{\alpha, n} \sum_{r=0,3} \left\{ \text{Tr}[I_n^\alpha t_{r0} Q_0] \text{Tr}[I_{-n}^\alpha t_{r0} Q_0] - \sum_{j=1}^3 \text{Tr}[I_n^\alpha t_{rj} Q_0] \text{Tr}[I_{-n}^\alpha t_{rj} Q_0] \right\} = -\frac{1}{2} O^{(\rho)} + \frac{1}{2} O^{(\sigma)}. \end{aligned} \quad (50)$$

Finally, we obtain

$$O^{(c)} \rightarrow O^{(c)} - YO^{(\rho)} + YO^{(\sigma)}. \quad (51)$$

It is worthwhile to mention that only particle-hole singlet and triplet interaction operators are involved in the renormalized particle-particle term. It is due to the different properties of I_n^α and L_n^α matrices under the transposition operation: $(I_n^\alpha)^T = I_{-n}^\alpha$, $(L_n^\alpha)^T = L_n^\alpha$.

E. One-loop renormalization group equations

Using Eqs. (44), (48) and (51), we find

$$\begin{aligned} S_{\text{int}}^{(\rho)} + S_{\text{int}}^{(\sigma)} + S_{\text{int}}^{(c)} &\rightarrow -\frac{\pi T}{4}(\Gamma_s - \Gamma_s Y - 3\Gamma_t Y - 2\Gamma_c Y)O^{(\rho)} \\ &- \frac{\pi T}{4}(\Gamma_t - \Gamma_s Y + \Gamma_t Y + 2\Gamma_c Y)O^{(\sigma)} - \frac{\pi T}{2}(\Gamma_c - \Gamma_s Y + 3\Gamma_t Y)O^{(c)} \end{aligned} \quad (52)$$

Next, since

$$Y = \frac{2}{g} \int \frac{d\mathbf{p}}{(2\pi)^d} D_p(0) = \frac{1}{\pi g} \ln L/l \quad (53)$$

we obtain the one-loop equations for the renormalization of the three interaction coupling constants

$$\frac{d\Gamma_s}{d \ln L} = -\frac{1}{\pi g}(\Gamma_s + 3\Gamma_t + 2\Gamma_c), \quad (54)$$

$$\frac{d\Gamma_t}{d \ln L} = -\frac{1}{\pi g}(\Gamma_s - \Gamma_t - 2\Gamma_c), \quad (55)$$

$$\frac{d\Gamma_c}{d \ln L} = -\frac{1}{\pi g}(\Gamma_s - 3\Gamma_t). \quad (56)$$

The right-hand sides of these equations are linear in the interaction couplings, since we have neglected the ladder resummation in the interaction propagators, as appropriate for weak couplings. The flow of the field renormalization constant z follows from the renormalization of Γ_s in view of the particle number conservation $d\Gamma_s/d \ln L = -dz/d \ln L$:

$$\frac{dz}{d \ln L} = -\frac{d\Gamma_s}{d \ln L} = \frac{1}{\pi g}(\Gamma_s + 3\Gamma_t + 2\Gamma_c). \quad (57)$$

Introducing $\gamma_{s,t,c} = \Gamma_{s,t,c}/z$ and taking into account the definition of $t = 2/\pi g$, we arrive at the linearized RG equations of the main text [see equations (70)-(74) below]:

$$\frac{d\gamma_s}{d \ln L} = -\frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c), \quad (58)$$

$$\frac{d\gamma_t}{d \ln L} = -\frac{t}{2}(\gamma_s - \gamma_t - 2\gamma_c), \quad (59)$$

$$\frac{d\gamma_c}{d \ln L} = -\frac{t}{2}(\gamma_s - 3\gamma_t). \quad (60)$$

F. Relation to the BCS Hamiltonian

Let us consider the interaction part of the Hamiltonian:

$$H_{\text{int}} = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' U(\mathbf{r} - \mathbf{r}') \bar{\psi}_\sigma(\mathbf{r}) \psi_\sigma(\mathbf{r}) \bar{\psi}_{\sigma'}(\mathbf{r}') \psi_{\sigma'}(\mathbf{r}') \quad (61)$$

In the BCS case (for example, for short-range attraction mediated by phonons),

$$U(\mathbf{r} - \mathbf{r}') = -\frac{\lambda}{\nu} \delta(\mathbf{r} - \mathbf{r}'),$$

where the thermodynamic density of states ν accounts for spin. According to Ref. [S2] the interaction parameters can be written as

$$\gamma_t = -\frac{F_t}{1 + F_t}, \quad \gamma_s = -\frac{F_s}{1 + F_s}, \quad \gamma_c = -F_c \quad (62)$$

where

$$F_t = -\frac{\nu}{2} \langle U(2k_F \sin(\theta/2)) \rangle_{FS}, \quad F_s = \nu U(q) + F_t, \quad F_c = -\frac{\nu}{4} \langle U(2k_F \sin(\theta/2)) \rangle_{FS} - \frac{\nu}{4} \langle U(2k_F \cos(\theta/2)) \rangle_{FS}. \quad (63)$$

Here $\langle \dots \rangle_{FS}$ denotes averaging over the Fermi surface. In the case of BSC Hamiltonian we find

$$F_t = \lambda/2, \quad F_s = -\lambda/2, \quad F_c = \lambda/2 \quad (64)$$

and

$$\gamma_t \approx -\lambda/2, \quad \gamma_s \approx \lambda/2, \quad \gamma_c = -\lambda/2. \quad (65)$$

Thus, for the BCS case (i.e. when neither screened nor unscreened Coulomb repulsion is taken into account), we get the following interaction parameters at the ultraviolet scale (which is given by Debye frequency ω_D in the case of phonon-induced superconductivity):

$$-\gamma_s = \gamma_t = \gamma_c = -\lambda/2.$$

As we will see in Sec. II below, this is precisely the relevant direction for the RG flow. When disorder is weak, $\omega_D \tau \gg 1$, the initial values for these couplings in Eqs. (58)-(60) are taken at the scale corresponding to the elastic scattering rate $1/\tau$. Then the Cooper interaction constant is renormalized at ballistic scales (between $1/\tau$ and ω_D) such that

$$-\gamma_{s,0} = \gamma_{t,0} = -\lambda/2, \quad \gamma_{c,0} = -\frac{\lambda/2}{1 - (\lambda/2) \ln \omega_D \tau} = \frac{1}{\ln T_c^{BCS} \tau} \quad (66)$$

where $T_c^{BCS} = \omega_D \exp(-2/\lambda)$.

The RG equations imply that there is the following perturbative correction to the BCS coupling due to interaction in the particle-hole channel

$$\gamma_c = \gamma_{c,0} - \frac{\lambda t_0}{2} \ln \frac{1}{T \tau}, \quad (67)$$

which is logarithmic in temperature. This leads to increase of the transition temperature:

$$\frac{\delta T_c}{T_c^{BCS}} = \frac{t_0 \lambda}{2 \gamma_{c,0}^2} \ln \frac{1}{T_c^{BCS} \tau} = \frac{t_0 \lambda}{2} \left(\ln \frac{1}{T_c^{BCS} \tau} \right)^3. \quad (68)$$

This perturbative calculation is justified in the regime $\delta T_c / T_c^{BCS} \ll 1$ which corresponds to the condition $t_0 \ll |\gamma_{c,0}|^3 / \lambda$. In order to neglect the renormalization of t due to weak localization disorder should be sufficiently weak, $t_0 \ll |\gamma_{c,0}|$ (see Sec. II). Therefore, the perturbative correction to T_c given by Eq. (68) is valid for $t_0 / |\gamma_{c,0}| \ll \min\{1, |\gamma_{c,0}|^2 / \lambda\}$.

In sufficiently dirty systems, $\omega_D \tau \ll 1$, the bare values of interactions constants in Eqs. (58)-(60) are taken at the scale corresponding to ω_D . In this case,

$$\frac{\delta T_c}{T_c^{BCS}} = \frac{2 t_0}{\lambda} \ln \frac{\omega_D}{T_c^{BCS}} = \frac{4 t_0}{\lambda^2}, \quad (69)$$

which holds for $t_0 \ll \gamma_{c,0}^2$. In the next section we will analyze the case of stronger disorder for which the perturbative treatment is insufficient and one should solve the full set of RG equations.

II. ANALYSIS OF RENORMALIZATION GROUP EQUATIONS

We consider first 2D systems of the orthogonal symmetry class at weak disorder ($g \gg 1$) where the localization and the multifractality effects are weak. Then we turn to 2D systems of the symplectic symmetry class in the metallic regime ($g \gg 1$) characterized by weak antilocalization. Finally, we address the vicinity of an Anderson transition (symplectic class in 2D and orthogonal class in 3D). In all these cases the multifractal character of wave functions in dirty systems can strongly enhance the superconducting transition temperature as compared to that of the clean system (usual BCS).

A. 2D, orthogonal symmetry class

The full set of RG equations for the orthogonal symmetry class reads:

$$\frac{dt}{d \ln L} = t^2 - \left(\frac{\gamma_s}{2} + 3 \frac{\gamma_t}{2} + \gamma_c \right) t^2, \quad (70)$$

$$\frac{d\gamma_s}{d \ln L} = -\frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c), \quad (71)$$

$$\frac{d\gamma_t}{d \ln L} = -\frac{t}{2}(\gamma_s - \gamma_t - 2\gamma_c), \quad (72)$$

$$\frac{d\gamma_c}{d \ln L} = -\frac{t}{2}(\gamma_s - 3\gamma_t) - 2\gamma_c^2, \quad (73)$$

$$\frac{d \ln z}{d \ln L} = \frac{t}{2}(\gamma_s + 3\gamma_t + 2\gamma_c). \quad (74)$$

Here, we remind, $t \ll 1$ is the dimensionless resistance (inverse dimensionless conductance, $t = 2/\pi g$), $\gamma_i = \Gamma_i/z \ll 1$ are interaction constants in the singlet (s), triplet (t) and Cooper (c) channels. In Eq. (70) the first term is due to weak localization. The second term is due to interaction [Altshuler-Aronov (AA) correction], it can be neglected within our accuracy. In Eqs. (71)–(73) only terms linear in interaction constants have been kept. The only exception is the Cooper term $-2\gamma_c^2$ in Eqs. (73) which is responsible for the BCS superconducting transition. The neglected terms are much smaller because they contain an additional small factor t . Below we shall use $y = \ln L$ where L is the running RG length scale. We measure lengths in units of the microscopic scale where the RG (70)–(74) starts.

Neglecting the AA contributions, we rewrite the set of equations (70)–(73) in the form

$$\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & -2 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2\gamma_c^2 \end{pmatrix}; \quad (75)$$

$$\frac{dt}{dy} = t^2. \quad (76)$$

Eigenvalues of the matrix in Eq. (75) are $\lambda = -4$, $\lambda' = 2, 2$ (not including the prefactor $-t/2$); the corresponding eigenvectors are as follows:

$$\lambda = -4 : \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad \lambda' = 2 : \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}. \quad (77)$$

If the γ_c^2 term is neglected, the solution of the linear system (75) approaches the eigenvector with $\lambda = -4$, i.e., $\gamma_s = -\gamma_t = -\gamma_c$. Let us expand the vector formed by γ_i in eigenvectors (77):

$$\begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix} = a \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \quad (78)$$

For convenience, we also present the inverted transformation:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1/6 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_t \\ \gamma_c \end{pmatrix}. \quad (79)$$

Transforming the set of equations (75) to the new variables a, b, c , we get

$$\frac{da}{dy} = 2ta - \frac{2}{3}(a - b + 2c)^2, \quad (80)$$

$$\frac{db}{dy} = -tb, \quad (81)$$

$$\frac{dc}{dy} = -tc - \frac{2}{3}(a - b + 2c)^2, \quad (82)$$

$$\frac{dt}{dy} = t^2. \quad (83)$$

Equations (80)-(83) are supplemented by the following initial conditions: $a(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$ and $t(0) = t_0$. Equation (83) yield the standard evolution of the resistance due to weak localization:

$$t(y) = \frac{t_0}{1 - t_0 y}. \quad (84)$$

Equation (81) can now be solved, yielding

$$b(y) = b_0(1 - t_0 y) \equiv b_0 \frac{t_0}{t}. \quad (85)$$

Since b decreases upon RG, it is not important and we neglect it in the future analysis.

Equations for the remaining two variables, a and c are coupled. If the quadratic term is neglected, then a increases and c decreases. This suggests that c can be neglected. This is confirmed by a more careful analysis which shows that, although on the very last interval of RG “time” y the variable c starts to increase and becomes of the same order as a (i.e. of order unity), this weakly affect the RG scale at which this happens (i.e. the temperature of the superconducting transition). Thus, we neglect c in what follows.

We can now easily solve the remaining equation for a . We assume the starting value a_0 to be negative (which means that there is attraction in the Cooper channel that is supposed to lead to the superconductivity), $a = -|a|$. This is in particular the case when $\gamma_{c,0}$ is the dominant coupling and $\gamma_{c,0} < 0$. The equation reads

$$\frac{d|a|}{dy} = 2t|a| + \frac{2}{3}a^2. \quad (86)$$

Solving this equation, we obtain

$$a(y) = - \left(\frac{t_0^2}{|a_0|t^2} + \frac{2t_0}{3t^2} - \frac{2}{3t} \right)^{-1}. \quad (87)$$

Let us analyze the obtained result. Let us first assume that $|a_0| \ll t_0$. Then the second term in brackets in the r.h.s. of (87) is small compared to the first one and can be neglected,

$$a^{-1}(y) = -\frac{1}{t} \left(\frac{t_0^2}{|a_0|t} - \frac{2}{3} \right). \quad (88)$$

With increasing RG scale y the resistance t increases together with the interaction a . If t reaches first unity, we get an insulator; if $a \sim 1$ happens first, we get a superconductor. It is easy to see that the second possibility (superconductivity) is realized if $|a_0| \gg t_0^2$. Then at the point of the transition to superconductivity we have a resistance

$$t_* \simeq \frac{3t_0^2}{2|a_0|} \ll 1. \quad (89)$$

The transition will then happen at

$$y_* \simeq \frac{1}{t_0} - \frac{1}{t_*} \quad (90)$$

i.e. at temperature

$$T_c^* \sim \exp \left\{ -\frac{2}{t_0} \left[1 - \frac{t_0}{t_*} \right] \right\}. \quad (91)$$

Here the factor 2 in the exponent originates from a translation of the length scale into energy (temperature). Under the above assumption $|a_0| \ll t_0$ the second term in square brackets in the exponential of (91) is just a small correction to the first one. By solving (82) with $b = 0$ and a given by Eq. (88), one finds that although $|c|$ decreases initially, eventually with increasing RG scale towards y_* it becomes of the order of unity: $|c(y_*)| \sim 1$. Therefore, to determine precise value of t_* one has to solve coupled equations for a and c (with $(b = 0)$).

The transition temperature (91) is much higher than the BCS temperature $T_{\text{BCS}} \sim e^{-1/|\gamma_{c,0}|}$, so that the superconductivity is strongly enhanced by disorder. The origin of the enhanced superconductivity is in the increase of $|a|$ governed by the eigenvalue $\lambda = -4$ of Eq. (77) which yields the eigenvalue $-(t/2) \times (-4) = 2t$ of the linear part of the system (75). This is nothing but the anomalous multifractal exponent $-\Delta_2$ for this symmetry class. (We have in mind the “weak multifractality” in 2D.) Therefore, the (multi)fractality is the source of the enhancement of the superconductivity.

It is worth mentioning that there is no such enhancement in the absence of interaction in particle-hole singlet and triplet channels. Indeed, the interplay of disorder and interaction in the renormalization of γ_c does not produce the term $t\gamma_c$ on the r.h.s. Eq. (73). Of course, if initially absent, the triplet and singlet amplitudes are generated due to the interplay of the Cooper amplitude and disorder, see Eq. (71) and (72). Furthermore, as we have seen in Sec. IF above, the BCS interaction in fact contributes to all interaction channels, so that even in the absence of electron-electron repulsion the bare values of γ_s and γ_t are non-zero.

If $|a_0| \ll t_0^2$, the resistance reaches unity before the interaction becomes strong, and the system is an insulator. Finally, if $|a_0| \gg t_0$, the disorder is not particularly important, and the transition temperature is given by usual clean BCS, $T_* \sim e^{-1/|\gamma_{c,0}|}$. (In the latter case neglecting b and c is not parametrically justified and leads to an incorrect numerical factor in the exponent.)

B. 2D, symplectic symmetry class, metallic regime

Now we consider the symplectic class, i.e. assume that the spin symmetry is broken for example, by sufficiently strong spin-orbit interaction. This leads to the following two modifications: (i) weak antilocalization rather than weak localization, and (ii) triplet interaction channel is suppressed and can be discarded. With these modifications, the system (70)–(73) becomes (cf. Ref. [S4])

$$\frac{dt}{dy} = -\frac{1}{2}t^2 - \left(\frac{\gamma_s}{2} + \gamma_c\right)t^2, \quad (92)$$

$$\frac{d\gamma_s}{dy} = -\frac{t}{2}(\gamma_s + 2\gamma_c), \quad (93)$$

$$\frac{d\gamma_c}{dy} = -\frac{t}{2}\gamma_s - 2\gamma_c^2. \quad (94)$$

Similar equations for the symplectic case of long-range Coulomb interaction have been derived in Ref. [S5]. The Altshuler-Aronov terms in (92) can again be neglected. Equation (92) then yields the standard antilocalization behavior,

$$t(y) = \frac{t_0}{1 + yt_0/2}. \quad (95)$$

The system of equations (92), (93) in the matrix form is:

$$\frac{d}{dy} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix} = -\frac{t}{2} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix} - \begin{pmatrix} 0 \\ 2\gamma_c^2 \end{pmatrix}. \quad (96)$$

Eigenvalues of the matrix in Eq. (96) are $\lambda = -1$, $\lambda' = 2$ (not including the prefactor $-t/2$); the corresponding eigenvectors are as follows:

$$\lambda = -1 : \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \quad \lambda' = 2 : \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (97)$$

If the γ_c^2 term is neglected, the solution of the linear system approaches the eigenvector with $\lambda = -1$, i.e., $\gamma_s = -\gamma_c$. As in the orthogonal case, we can expand the vector formed by γ_i in eigenvectors

$$\begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix} = a \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}. \quad (98)$$

The inverted transformation is given as:

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} \gamma_s \\ \gamma_c \end{pmatrix}. \quad (99)$$

Transforming the set of equations (98) to the new variables a, c , we find

$$\frac{da}{dy} = \frac{t}{2}a - \frac{4}{3}(a+c)^2, \quad (100)$$

$$\frac{dc}{dy} = -tc - \frac{2}{3}(a+c)^2. \quad (101)$$

Equations for two variables, a and c are coupled. If the quadratic term is neglected, then a increases and c decreases. At the later stage of RG the quadratic terms leads to enhancement of c . This suggests that c can be neglected for qualitative analysis of RG equations (100)-(101). We thus neglect c and keep only a (fully analogously to what we have done in the orthogonal case). The resuting equation for a reads

$$\frac{d|a|}{dy} = \frac{t}{2}|a| + \frac{4}{3}a^2. \quad (102)$$

We solve this equation with the result

$$a = \frac{1}{t} \left(\frac{1}{a_0 t_0} + \frac{4}{3t_0^2} - \frac{4}{3t^2} \right)^{-1}. \quad (103)$$

The new (different from usual BCS) behavior emerges under the condition $a_0 \ll t_0$. Then the condition $a \sim 1$ yields

$$t_* \simeq 2(a_0 t_0 / 3)^{1/2} \ll t_0. \quad (104)$$

By solving Eq. (101) with a given by Eq. (103), we find that although $|c|$ decreases initially, eventually with increasing RG scale it reaches a : $c \sim a$ at $t = t_*$. Therefore, to determine precise value of t_* one has to solve coupled equations for a and c .

Equation (104) yields the transition temperature

$$T_c^* \sim e^{-2y_*} \sim e^{-4/t_*} \sim \exp \left\{ -\mathcal{C} / (a_0 t_0)^{1/2} \right\}, \quad (105)$$

much higher than $T_{\text{BCS}} \sim e^{-1/|\gamma_{c,0}|}$. The BCS behavior is restored (up to corrections) at $a_0 \gg t_0$. The constant \mathcal{C} is of the order of unity and depend on ration c_0/a_0 .

As in the orthogonal case, the source of the enhancement of the superconducting temperature is in the first term on the r.h.s. of eq. (102). The eigenvalue $t/2$ is the anomalous multifractal exponent $-\Delta_2$ for the symplectic symmetry class. Therefore, also in this case the (multi)fractality is the source of the enhancement of the superconductivity. This enhancement is less efficient than in the orthogonal case for two reasons, because of antilocalizing behavior that leads to decrease of t and therefore weakening of multifractality.

C. System at or near Anderson transition

1. Exactly at criticality

We now consider a system at the Anderson transition point. This may be 2D or 3D symplectic class system, or 3D orthogonal class. In all the cases after the first (fast) part of the RG evolution, where the Cooper term is assumed to be not important yet, the coupling constant γ_s and (in the orthogonal case) γ_t “adjust” to γ_c according to the

$\gamma_s = -\gamma_t = -\gamma_c$ (orthogonal) or $\gamma_s = -\gamma_c$ (symplectic). So, the main part of evolution can be described by a single equation, in analogy with Eqs. (86) and (102):

$$\frac{d\gamma}{dy} = -\Delta_2\gamma - a\gamma^2, \quad a \sim 1 \quad (106)$$

The superconductivity will take place if $\gamma_0 < 0$. We also note that $\Delta_2 < 0$. In particular, $\Delta_2 = -1.7 \pm 0.05$ for the 3D orthogonal-class [S6] and $\Delta_2 = -0.344 \pm 0.004$ for the 2D symplectic-class Anderson transitions [S7]. If $|\gamma_0| \ll 1$, the second term in the r.h.s. of (106) is not important for the evaluation of the leading behavior of the transition temperature. Keeping only the first term, we find

$$T_c^* \sim \exp\{-dy_*\} \sim |\gamma_0|^{d/|\Delta_2|}. \quad (107)$$

The factor d in the exponential, where d stands for the spatial dimensionality, translates length in temperature. We see that at the Anderson transition point the enhancement of the superconducting transition temperature becomes even stronger than in 2D: the transition temperature is now a power-law (rather than exponential) function of the coupling constant γ_0 . Equation (107) agrees with the result of Ref. [S3].

2. Slightly off criticality: Insulating side

In the above consideration we assumed that the system is exactly at the Anderson transition point. Let us analyze what happens if the system is slightly off criticality. Then, we need to add the following equation to Eq. (106):

$$\frac{dt}{dy} = \frac{1}{\nu}(t - t_c) + \eta\gamma. \quad (108)$$

Here, we take in account that the presence of interaction drives the system away from the non-interacting critical point. We note that the correlation length exponent $\nu > 0$. In particular, $\nu = 1.57 \pm 0.02$ for the 3D orthogonal-class [S8] and $\nu = 2.746 \pm 0.009$ for the 2D symplectic-class Anderson transitions [S9].

Neglecting the term of the second order in γ , the system of Eqs. (106) and (108) can be solved for $|\Delta_2|\nu \neq 1$ as

$$t = \tilde{t} + \frac{\eta\nu}{|\Delta_2|\nu - 1}\gamma, \quad \tilde{t} = t_c + (\tilde{t}_0 - t_c)e^{y/\nu}, \quad \gamma = \gamma_0 e^{|\Delta_2|y}, \quad \tilde{t}_0 = t_0 - \frac{\eta\nu\gamma_0}{|\Delta_2|\nu - 1}. \quad (109)$$

In the special case $|\Delta_2|\nu = 1$, we find

$$\tilde{t} = t - \eta\gamma_0\gamma y, \quad \tilde{t}_0 = t_0. \quad (110)$$

Therefore, the presence of the term $\eta\gamma$ in Eq. (108) indicates that the proper scaling variable is \tilde{t} rather than t . In what follows we shall omit ‘tilde’ sign.

Up to the scale of the localization length

$$\xi \sim |t_0 - t_c|^{-\nu} \quad (111)$$

the RG will proceed as at the critical point. So, there are two possibilities. If $L_* < \xi$, where $L_* \sim |\gamma_0|^{-1/|\Delta_2|}$ is the length scale where the superconducting transition at criticality takes place (Sec. IIC 1), then the transition temperature is not affected by detuning. In the opposite case, the localization takes place first, and there is no superconductivity. Thus, the condition ($\delta_\xi \propto \xi^{-d}$)

$$\delta_\xi \sim T_c^*, \quad (112)$$

where T_c^* is given by Eq. (107), is the condition of the superconductor-insulator transition (at zero temperature). It is worth noting that Ref. [S3] argues that superconducting state with $T_c \ll T_c^*$ persists further in the localized regime ($\delta_\xi \gg T_c^*$) due to Mott-type rare configurations. Our RG approach (at least in its present form) is not sufficient to explore this possibility.

3. Slightly off criticality: Metallic side

As in Sec. IIC 2, we have the length scale ξ given by Eq. (111) but it now has a meaning of the correlation length. Below this scale the system is at criticality, above this scale it becomes a metal. As on the localized side, if $\delta_\xi \ll T_*$, the detuning from criticality is immaterial, and the transition temperature is given by T_* , Eq. (107). In the opposite case, the result depend on whether we are in 2D symplectic case or in 3D (4D, ...) orthogonal class.

4. 3D, 4D, ..., orthogonal class

After the first step of RG (up to the scale ξ) the Cooper-channel interaction constant takes the value

$$\tilde{\gamma}_0 = \gamma_0 \xi^{|\Delta_2|}. \quad (113)$$

After this the RG proceeds according to the usual BCS. The total RG scale y where the coupling becomes of order unity (and thus the transition happens) is

$$y_*^{(\xi)} = \ln \xi + \frac{c}{|\gamma_0| \xi^{|\Delta_2|}}, \quad (114)$$

where $c \sim 1$. Thus, the transition temperature is

$$T_*^{(\xi)} \sim \frac{1}{\xi^d} \exp \left\{ -\frac{cd}{|\gamma_0| \xi^{|\Delta_2|}} \right\} = \delta_\xi \exp \left\{ -cd \left(\frac{\delta_\xi}{T_*} \right)^{|\Delta_2|/d} \right\}. \quad (115)$$

It is easy to see that this intermediate regime correctly matches results for two regimes between which it is located: at $\delta_\xi \sim T_*$ we have $T_*^{(\xi)} \sim T_*$, and at $\delta_\xi \sim 1$ we have $T_*^{(\xi)} \sim T_{\text{BCS}}$.

5. 2D, symplectic class

After the first (Anderson-critical) step of RG the interaction constant is given by the same formula (113) as in 3D. The difference is on the second step: now we have to apply the formula for a 2D symplectic metal (105) with $t_0 \sim 1$. This yields

$$T_*^{(\xi)} \sim \frac{1}{\xi^2} \exp \left\{ -\frac{C}{|\gamma_0|^{1/2} \xi^{|\Delta_2|/2}} \right\} = \delta_\xi \exp \left\{ -C \left(\frac{\delta_\xi}{T_*} \right)^{|\Delta_2|/4} \right\}. \quad (116)$$

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